

RIGIDITY INDEX PRESERVATION OF REGULAR HOLONOMIC D-MODULES UNDER FOURIER TRANSFORM

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ABSTRACT. This paper shows algebraically that the Fourier transform preserves the rigidity index of irreducible regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$ -modules.

1. INTRODUCTION

Riemann showed in 1857 that the local system of the hypergeometric equation can be reconstructed up to isomorphism from the knowledge of the local monodromies around its singular points 0, 1 and ∞ , by analytic continuation of the solutions of the equation around the singular points. In modern terminology, local systems on a projective smooth connected curve X over \mathbb{C} , with singularities on a nonempty finite subset of X , satisfying the aforementioned condition are called physically rigid. In [4] Katz gave necessary and sufficient conditions for physical rigidity of local systems on the Riemann sphere, based upon on a cohomological numerical index. He showed that in characteristic $p > 0$, the Fourier transform preserves this index when the local system is a perverse sheaf that does not have punctual support nor does its Fourier transform (cf. [4] Theorem 3.0.2). Moreover he conjectured that "it should be true that Fourier transform preserves the index of rigidity in the \mathcal{D} -module context" (cf. [4], p. 10). This conjecture was proved by S. Bloch and H. Esnault in [1]. A different proof is given in this paper, when the \mathcal{D} -module is regular holonomic localized at infinity (Theorem 5.1).

The paper is divided into five sections. The first section reviews some results on rigidity. The second extends the notion of rigid local systems to the context of holonomic $\mathcal{D}_{\mathbb{P}^1}$ -modules. The third recalls the notion of Fourier transform and computes the rigidity index of the Fourier transform of irreducible regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$ -modules. The fourth translates the germs of holonomic \mathcal{D} -modules on the equivalent category of pairs of vector spaces. These equivalences are used in last section to show the preservation of the rigidity index referred above (Theorem 5.1). General references for this paper are [2, 3, 6, 10].

2. RIGIDITY INDEX

In [4] Katz gives the following necessary and sufficient condition for the physical rigidity of local systems on \mathbb{P}^1 .

Theorem 2.1. ([4], Theorem 1.1.2) *Let Σ be a non empty finite subset of \mathbb{P}^1 , $U \doteq \mathbb{P}^1 \setminus \Sigma$, $j : U^{an} \hookrightarrow (\mathbb{P}^1)^{an}$ the open inclusion and \mathcal{L} an irreducible local system on U^{an} of rank $n \geq 1$. Then \mathcal{L} is physically rigid if and only if $\chi((\mathbb{P}^1)^{an}, j_* \mathcal{E}nd(\mathcal{L})) = (2 - k)n^2 + \sum_i \dim Z(A_i) = 2$, where $Z(A_i) \doteq \{A \in \text{End}_{\mathbb{C}}(\mathcal{L}_{s_i}) \mid AA_i = A_i A\}$, $k + 1 = \#\Sigma$ and A_i is the monodromy of \mathcal{L} at the point $s_i \in \Sigma$.*

In order to extend the notion of rigidity to the context of holonomic $\mathcal{D}_{\mathbb{P}^1}$ -modules, one starts to recall the notion of minimal extension.

Definition 2.2. Let \mathcal{M} be a holonomic \mathcal{D} -module on a Riemann surface X and $\Sigma \subset X$ a finite set. One says that a holonomic \mathcal{D} -module \mathcal{N} on X is a **minimal extension** of \mathcal{M} along Σ and denote it \mathcal{M}_{min} if:

- i) $\mathcal{O}_X[*\Sigma] \otimes_{\mathcal{O}_X} \mathcal{M} = \mathcal{O}_X[*\Sigma] \otimes_{\mathcal{O}_X} \mathcal{N}$,
- ii) \mathcal{M} has neither nonzero submodules nor nonzero quotients with support on a subset of Σ .

It is well known that if \mathcal{M} is a regular holonomic \mathcal{D}_X -module with singularities on Σ , its local system $\mathcal{L} \doteq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})|_{X \setminus \Sigma}$ satisfies the following identity

$$\mathrm{DR}((\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{M}[*\Sigma]))_{min}) = j_* \mathcal{E}nd(\mathcal{L}).$$

Taking this into account and Theorem 2.1, one is led to give the following definition.

Definition 2.3 (Rigidity index). Let \mathcal{M} be an irreducible holonomic $\mathcal{D}_{\mathbb{P}^1}$ -module and Σ the set of its singular points. Set $\mathrm{rig}(\mathcal{M})$ the invariant

$$\chi\left(\mathbb{P}^1, \mathrm{DR}\left((\mathcal{E}nd_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{M}[*\Sigma]))_{min}\right)\right)$$

and call it the **rigidity index** of \mathcal{M} .

Follows three propositions on minimal extension which will be used on this paper.

Proposition 2.4. ([8], Corollary 2.7.4) *If \mathcal{M} is a holonomic \mathcal{D} -module on a Riemann surface X and $\Sigma \subset X$ a finite set, then the \mathcal{D} -modules \mathcal{M} and $\mathcal{M}[*\Sigma]$ have the same minimal extension along Σ , i.e. $\mathcal{M}_{min} = \mathcal{M}[*\Sigma]_{min}$.*

Proposition 2.5. ([8], Theorem 2.7.6) *Let \mathcal{M} be a holonomic \mathcal{D} -module on a Riemann surface X and $\Sigma \subset X$ a finite set, then the minimal extension of \mathcal{M} along Σ exists and is given by*

$$\mathcal{M}_{min} = \left((\mathcal{M}/\mathrm{H}_{[\Sigma]}(\mathcal{M}))^* / \mathrm{H}_{[\Sigma]} \left((\mathcal{M}/\mathrm{H}_{[\Sigma]}(\mathcal{M}))^* \right) \right)^*,$$

where $\mathrm{H}_{[\Sigma]}(\mathcal{M})(U) \doteq \{s \in \mathcal{M}(U) \mid \exists x_i \in \Sigma \exists k_i \in \mathbb{N} : (x - x_i)^{k_i} s = 0\}$, for each open set $U \subsetneq \mathbb{P}^1$ and local coordinate x on U .

As for holonomic \mathcal{D} -modules on a Riemann surface X with singularities on Σ $\mathrm{supp}(\mathcal{M}) \subset \Sigma$, iff \mathcal{M} coincides with its algebraic support on Σ , i.e. $\mathcal{M} = \mathrm{H}_{[\Sigma]}(\mathcal{M})$, cf. [8] Lemma 2.7.8, the notion of minimal extension at the level of germs is defined as follows.

Definition 2.6. Let \mathcal{M} be a holonomic $\mathcal{D}_x \doteq \mathbb{C}\{x\}\langle\partial_x\rangle$ (resp. $\widehat{\mathcal{D}}_x \doteq \mathbb{C}[[x]]\langle\partial_x\rangle$)-module. One says that a holonomic \mathcal{D}_x (resp. $\widehat{\mathcal{D}}_x$)-module \mathcal{N} is a **minimal extension** of \mathcal{M} and denote it \mathcal{M}_{min} if:

- i) $\mathcal{M}[x^{-1}] = \mathcal{N}[x^{-1}]$,
- ii) \mathcal{M} has neither nonzero submodules nor nonzero quotients coinciding with its algebraic support on 0.

Proposition 2.7. ([8], Theorem 2.7.11) *The minimal extension commutes with the formalized, that is, if \mathcal{M} is a holonomic \mathcal{D}_x -module, then $\widehat{\mathcal{M}_{min}} \simeq (\widehat{\mathcal{M}})_{min}$.*

3. FOURIER TRANSFORM

The notion of Fourier transform is built on the concept of *twisted modules*, cf. [3] p. 38. Let us recall this notion. Let R be a ring, M a left R -module and σ an automorphism of R . M_σ is the left R -module M with the new action $a \bullet m \doteq \sigma(a)m$, for $a \in R$ and $m \in M$. A routine calculation shows that M_σ is a left R -module and σ defines a functor from the category of left R -modules into itself. M_σ is called the *twisted module* of M by σ . Let us apply this construction to $A_1 \doteq \mathbb{C}[x]\langle \partial_x \rangle$ (resp. $A_1[x^{-1}]$ or \mathcal{D} or $\widehat{\mathcal{D}}$) to define the Fourier transform (resp. the inversion).

Definition 3.1. The Fourier transform and the inversion are, respectively, the following automorphisms:

$$\begin{array}{ccc} \mathcal{F} : \mathbb{C}[x]\langle \partial_x \rangle & \longrightarrow & \mathbb{C}[x]\langle \partial_x \rangle, & \mathcal{I} : \mathbb{C}[x, x^{-1}]\langle \partial_x \rangle & \longrightarrow & \mathbb{C}[x, x^{-1}]\langle \partial_x \rangle. \\ x & \longmapsto & -\partial_x & x & \longmapsto & x^{-1} \\ \partial_x & \longmapsto & x & \partial_x & \longmapsto & -x^2 \partial_x \end{array}$$

To extend the notion of Fourier transform from the context of holonomic A_1 -modules to the context of holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}] \doteq \mathcal{D}_{\mathbb{P}^1}^{\text{ana}}[*\{\infty\}]$ -modules, let us recall the correspondence between holonomic A_1 , $\mathcal{D}_{\mathbb{P}^1}^{\text{alg}}[*\{\infty\}]$ and $\mathcal{D}_{\mathbb{P}^1}^{\text{ana}}[*\{\infty\}]$ -modules. Let M be a holonomic A_1 -module with singularities on $S = \{\gamma_1, \dots, \gamma_k\} \subset \mathbb{C}$, where $k \geq 1$. Let M' be the holonomic $A_1[x^{-1}]$ -module $M[x^{-1}]_{\mathcal{I}}$. This two modules generate the holonomic $\mathcal{D}_{\mathbb{P}^1}^{\text{alg}}[*\{\infty\}]$ -module \mathcal{M}^{alg} defined by $\mathcal{M}^{\text{alg}}(\mathbb{C}) = M$ and $\mathcal{M}^{\text{alg}}(\mathbb{P}^1 \setminus \{\infty\}) = M'$ and vice versa. The correspondence between $\mathcal{M}^{\text{ana}} \doteq \mathcal{M}^{\text{alg}} \otimes_{\mathcal{O}_{\mathbb{P}^1}^{\text{alg}}} \mathcal{O}_{\mathbb{P}^1}^{\text{ana}}$ and \mathcal{M}^{alg} is done by GAGA, cf. [7] chap. I §4. The holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$ -module $\mathcal{M}_{\mathcal{F}}$ built this way from $M_{\mathcal{F}}$ is called the *Fourier transform* of $M \doteq \mathcal{M}^{\text{ana}}$. In the special case when all singularities $\Sigma \doteq \{\gamma_0 = \infty\} \cup S$ of \mathcal{M} are regular, $\mathcal{M}_{\mathcal{F}}$ has a regular singularity at 0 and one (possibly irregular) at ∞ , cf. [7] chap. V §1. Now follows the notation used to compute $\text{rig}(\mathcal{M}_{\mathcal{F}})$.

Notation 3.2. Let \mathcal{M} be a regular holonomic $\mathcal{D}_{\mathbb{P}^1}$ -module with singularities on Σ .

- i) $(\widehat{\mathcal{N}}, \widehat{\nabla})$ denotes the formalized at ∞ of (\mathcal{N}, ∇) of the usual meromorphic connexion equivalent to $\mathcal{M}_{\mathcal{F}}[*\{0, \infty\}]$, which has the decomposition of Turrittin

$$(1) \quad (\widehat{\mathcal{N}}, \widehat{\nabla}) \simeq \bigoplus_{i=1}^k \widehat{\mathcal{E}}^{\varphi_i} \otimes (\widehat{\mathcal{R}}_i, \widehat{\nabla}_i)$$

and $(\widehat{\mathcal{R}}_i, \widehat{\nabla}_i)$ are regular meromorphic connexions, cf. Turrittin [11], Levelt [5], and [8] Theorem 1.9.5 and Lemma 1.96;

- ii) T_i denotes the monodromy of $(\widehat{\mathcal{R}}_i, \widehat{\nabla}_i)$ and n_i the dimension of $\widehat{\mathcal{R}}_i$;
- iii) T denotes the monodromy at 0 of the local system $\mathcal{H}om_{\mathcal{D}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{M}_{\mathcal{F}})|_{\mathbb{C}^*}$.

Theorem 3.3. *If \mathcal{M} is a regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$ -module with singularities on Σ , the rigidity index of $\mathcal{M}_{\mathcal{F}}$ is given by*

$$(2) \quad \text{rig}(\mathcal{M}_{\mathcal{F}}) = \dim Z(T) + \sum_{i=1}^k \dim Z(T_i) + \sum_{i=1}^k n_i^2 - \left(\sum_{i=1}^k n_i \right)^2.$$

Proof. Let \mathcal{E} be the $\mathcal{D}_{\mathbb{P}^1}$ -module $\mathcal{E}nd_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{M}_{\mathcal{F}}[*\{0, \infty\}])$, \mathcal{F}^\bullet the de Rham complex $\text{DR}(\mathcal{E}_{\min})$ on \mathbb{P}^1 and $j : \mathbb{C}^* \hookrightarrow \mathbb{P}^1$ the open inclusion. One has the short exact

sequence

$$(3) \quad 0 \longrightarrow j_! j^{-1} \mathcal{F}^\bullet \xrightarrow{\eta} \mathcal{F}^\bullet \longrightarrow \text{coker } \eta \longrightarrow 0,$$

which yields the identity:

$$\chi(\mathbb{P}^1, \mathcal{F}^\bullet) = \chi(\mathbb{P}^1, j_! j^{-1} \mathcal{F}^\bullet) + \chi(\mathbb{P}^1, \text{coker } \eta).$$

Set $\mathcal{L} = h^0(j^{-1} \mathcal{F}^\bullet)$, hence:

$$\begin{aligned} \chi(\mathbb{P}^1, j_! j^{-1} \mathcal{F}^\bullet) &= (2 - 2) \text{rang } \mathcal{L} = 0, \\ \chi(\mathbb{P}^1, \text{coker } \eta) &= \chi(\mathbb{P}^1, (\text{coker } \eta)_0) + \chi(\mathbb{P}^1, (\text{coker } \eta)_\infty), \end{aligned}$$

because $\{0, \infty\}$ are the only singularities of \mathcal{F}^\bullet , thus:

$$(4) \quad \chi(\mathbb{P}^1, \mathcal{F}^\bullet) = \chi(\mathbb{P}^1, (\text{coker } \eta)_0) + \chi(\mathbb{P}^1, (\text{coker } \eta)_\infty).$$

To compute $\chi(\mathbb{P}^1, (\text{coker } \eta)_0)$ one takes a disk $D \subset \mathbb{C}$ centered at 0 and the inclusion $i : D^* \hookrightarrow D$. Set $\mathcal{G}^\bullet = \text{DR}(\mathcal{E}_{\min}|_D)$. As $\mathcal{E}_{\min}|_D$ is a regular holonomic \mathcal{D}_D -module on D , $\text{DR}(\mathcal{E}_{\min}|_D) = i_* \text{End}(\mathcal{L}')$, where $\mathcal{L}' = \mathcal{H}om_{\mathcal{D}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{M}_{\mathcal{F}})|_{D^*}$. \mathcal{G}^\bullet is a perverse complex on D and gives rise to the short exact sequence

$$(5) \quad 0 \longrightarrow i_! i^{-1} \mathcal{G}^\bullet \xrightarrow{\eta|_D} \mathcal{G}^\bullet \longrightarrow (\text{coker } \eta)|_D \longrightarrow 0,$$

which yields

$$\chi(D, \mathcal{G}^\bullet) = \chi(D, i_! i^{-1} \mathcal{G}^\bullet) + \chi(D, \text{coker } \eta|_D).$$

Since $h^0(i^{-1} \mathcal{G}^\bullet) = \mathcal{E}nd(\mathcal{L}')$, one has $\chi(D, i_! i^{-1} \mathcal{G}^\bullet) = (2 - 2) \text{rang } \mathcal{E}nd(\mathcal{L}') = 0$, because D^* is homotopic to S^1 , therefore $\chi(D, \mathcal{G}^\bullet) = \chi(D, (\text{coker } \eta)_0)$. The exact sequence (5) implies that $(\mathcal{G}^\bullet)_0 = (\text{coker } \eta)_0$. Moreover $(\mathcal{G}^\bullet)_0 = \{M \in \text{End}(E) \mid T_{\mathcal{E}nd(\mathcal{L}')} (M) = M\}$, where $E = h^0(D \setminus \mathbb{R}^+, \mathcal{L}')$, $T_{\mathcal{E}nd(\mathcal{L}')} = \text{ad}_T$ and T is the monodromy of \mathcal{L}' at 0, thus $(\mathcal{G}^\bullet)_0 = \{M \in \text{End}(E) \mid TM = MT\} \doteq Z(T)$. By Mayer-Vietoris one shows that $\chi(\mathbb{P}^1, (\text{coker } \eta)_0) = \chi(D, \text{coker } \eta|_D)$, hence

$$(6) \quad \chi(\mathbb{P}^1, (\text{coker } \eta)_0) = \dim Z(T).$$

Now one computes $\chi(\mathbb{P}^1, (\text{coker } \eta)_\infty)$. As $(j_! j^{-1} \mathcal{F}^\bullet)_\infty = 0$, the exact sequence (3) implies that $(\mathcal{F}^\bullet)_\infty \simeq (\text{coker } \eta)_\infty$. On the other hand $\chi(\mathbb{P}^1, (\text{DR}(\mathcal{E}_{\min}))_\infty) = \chi((\mathcal{E}_{\min})_\infty, (\mathcal{O}_{\mathbb{P}^1})_\infty)$, therefore by definition of irregularity

$$(7) \quad \chi(\mathbb{P}^1, (\text{DR}(\mathcal{E}_{\min}))_\infty) = \chi(\widehat{(\mathcal{E}_{\min})_\infty}, (\widehat{\mathcal{O}_{\mathbb{P}^1}})_\infty) - i((\mathcal{E}_{\min})_\infty).$$

Owing to Lemma 3.4 and Proposition 2.7

$$(8) \quad \chi(\widehat{(\mathcal{E}_{\min})_\infty}, (\widehat{\mathcal{O}_{\mathbb{P}^1}})_\infty) = \sum_{i=1}^k \dim Z(T_i).$$

Furthermore, by Lemma 3.4

$$(9) \quad i((\mathcal{E}_{\min})_\infty) = \left(\sum_{i=1}^k n_i \right)^2 - \sum_{i=1}^k n_i^2.$$

The identity (2) is now an immediate consequence of (4), (6), (7), (8) and (9). \square

Lemma 3.4. *Let $(\widehat{\mathcal{N}}, \widehat{\nabla})$ be the formalized at ∞ of the (usual) meromorphic connection associated to the Fourier transform of a regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$ -module with singularities on Σ . Assume that $\varphi_1 = 0$ on the Turrittin decomposition of $(\widehat{\mathcal{N}}, \widehat{\nabla})$, notice that $(\widehat{\mathcal{R}}_1, \widehat{\nabla}_1)$ might be 0, then:*

$$\text{i) } \chi(\widehat{\mathcal{N}}_{\min}, \mathbb{C}[[x]]) = \dim\{e \mid T_1 e = e\},$$

$$\begin{aligned} \text{ii)} \quad & \chi(\mathcal{E}nd_{\widehat{\mathcal{O}}_x}(\widehat{\mathcal{N}})_{min}, \mathbb{C}[[x]]) = \sum_{i=1}^k \dim Z(T_i), \\ \text{iii)} \quad & i(\mathcal{E}nd_{\mathcal{O}_x}(\mathcal{N})) = \left(\sum_{i=1}^k n_i \right)^2 - \sum_{i=1}^k n_i^2. \end{aligned}$$

Proof. i) For each term from decomposition (1) either $i = 1$ or $i > 1$.

If $i > 1$, the holonomic $\widehat{\mathcal{D}}_x$ -module $\widehat{\mathcal{N}}_i \doteq \widehat{\mathcal{E}}^{\varphi_i} \otimes (\widehat{\mathbf{R}}_i, \widehat{\mathbf{V}}_i)$ has no regular component, therefore $\widehat{\mathcal{N}}_i \simeq \widehat{\mathcal{N}}_i[x^{-1}]$ (cf. [6] Theorem 6.3.1). This implies that the multiplication by x is bijective so $H_{[0]}(\widehat{\mathcal{N}}_i) = 0$ and $H_{[0]}(\widehat{\mathcal{N}}_i^*) = 0$, hence $(\widehat{\mathcal{N}}_i)_{min} = \widehat{\mathcal{N}}_i[x^{-1}]$ by Proposition 2.5. Thanks to these isomorphisms one has $\chi(\widehat{\mathcal{N}}_{min}, \mathbb{C}[[x]]) = \chi((\widehat{\mathcal{N}}_1)_{min}, \mathbb{C}[[x]])$, because $\chi(\widehat{\mathcal{N}}_i[x^{-1}], \mathbb{C}[[x]]) = 0$ for $i > 1$.

If $i = 1$, choosing a base and a coordinate system one has the isomorphism

$$(\widehat{\mathbf{R}}_1, \widehat{\mathbf{V}}_1) \simeq \left(\mathbb{C}[[x]]^n, x \frac{d}{dx} - A_1 \right), n = \dim \widehat{\mathbf{R}}_1.$$

By Proposition 2.4, $\widehat{\mathcal{N}}_1$ and $\widehat{\mathcal{N}}_1[x^{-1}]$ have the same minimal extension, so one computes the minimal extension of the later. To do this, take a meromorphic change of base, which transforms A_1 in the constant matrix, J , in the Jordan canonical form

$$(\widehat{\mathbf{R}}_1[x^{-1}], \widehat{\mathbf{V}}_1) \simeq \bigoplus_{j=1}^m \left(\mathbb{C}[[x]]^{n_j}, x \frac{d}{dx} - J_j \right),$$

with $n_1 + \dots + n_m = n$ and J_j the Jordan blocs of J . If α_j is the eigenvalue of the Jordan block J_j , then one has the isomorphism

$$\widehat{\mathcal{N}}_1[x^{-1}] \simeq \bigoplus_{j=1}^m \mathbb{C}[[x]][x^{-1}] \langle \partial_x \rangle / \mathbb{C}[[x]][x^{-1}] \langle \partial_x \rangle \cdot (x \partial_x - \alpha_j)^{n_j}.$$

If $\alpha_j \notin \mathbb{Z}$, the Bernstein polynomial $b(x)$ of $\widehat{\mathcal{N}}_{1j} \doteq \mathbb{C}[[x]] \langle \partial_x \rangle / (x \partial_x - \alpha_j)^{n_j}$ is $(x - \alpha_j)^{n_j}$, therefore for each $k \in \mathbb{N}$, $b(k) \neq 0$, thus the multiplication by x

$$\mathbb{C}[[x]] \langle \partial_x \rangle / (x \partial_x - \alpha_j)^{n_j} \xrightarrow{x \cdot} \mathbb{C}[[x]] \langle \partial_x \rangle / (x \partial_x - \alpha_j)^{n_j}$$

is a bijective map (cf. [6] Lemma 4.2.7), i.e. $\widehat{\mathcal{N}}_{1j}$ is a meromorphic connexion. In particular one has $H_{[0]}(\widehat{\mathcal{N}}_{1j}) = 0$ and $H_{[0]}((\widehat{\mathcal{N}}_{1j})^*) = 0$, thus $(\widehat{\mathcal{N}}_{1j})_{min} = \widehat{\mathcal{N}}_{1j}$. As $\chi(\widehat{\mathcal{N}}_{1j}[x^{-1}], \mathbb{C}[[x]]) = 0$ for each $\alpha_j \notin \mathbb{Z}$, these equalities lead to:

$$\chi(\widehat{\mathcal{N}}_{min}, \mathbb{C}[[x]]) = \sum_{\alpha_j \in \mathbb{Z}} \chi((\widehat{\mathcal{N}}_{1j})_{min}, \mathbb{C}[[x]]).$$

If $\alpha_j \in \mathbb{Z}$, one can assume $\alpha_j = 0$, because after the change of base $B = x^{-\alpha_j} \mathbb{1}$, the matrix J_j is transformed into

$$BJ_j B^{-1} + x \partial B / \partial x B^{-1} = J_j - \alpha_j \mathbb{1}.$$

In this case $\widehat{\mathcal{N}}_j = \mathbb{C}[[x]] \langle \partial_x \rangle / (x \partial_x)^{n_j}$ and therefore $(\widehat{\mathcal{N}}_j)_{min} = \mathbb{C}[[x]] \langle \partial_x \rangle / R_j$, where $R_j = \partial_x (x \partial_x)^{n_j - 1}$. As $\{1, \log x, \dots, \log^{n_j - 1} x\}$ is a solution base of the differential equation $R_j y = 0$, the kernel of R_j in $\mathbb{C}[[x]]$ is $\langle 1 \rangle$, hence $\dim \ker R_j = 1$. Given that $\frac{1}{(l+1)^{n_j}} x^{l+1}$ is a solution of the differential equation $R_j y = x^l$, $\dim \text{coker } R_j = 0$.

Altogether $\chi(\widehat{\mathcal{N}}_{min}, \mathbb{C}[[x]]) = \#\{J_j \mid \alpha_j \in \mathbb{Z}\}$, that is $\chi(\widehat{\mathcal{N}}_{min}, \mathbb{C}[[x]]) = \dim\{e \mid T_1 e = e\}$.

ii) Take the decomposition of Turruttin (1) $(\widehat{\mathcal{N}}, \widehat{\nabla}) \simeq \bigoplus_{i=1}^k \widehat{\mathcal{E}}^{\varphi_i} \otimes \widehat{\mathbf{R}}_i$. Since:

$$\begin{aligned}
 (10) \quad & \left(\text{Hom}_{\widehat{\mathcal{O}}_x} \left(\bigoplus_{i=1}^k \widehat{\mathcal{E}}^{\varphi_i} \otimes \widehat{\mathbf{R}}_i, \bigoplus_{i=1}^k \widehat{\mathcal{E}}^{\varphi_i} \otimes \widehat{\mathbf{R}}_i \right), \widehat{\nabla} \right) \simeq \\
 & \simeq \bigoplus_{i=1}^k \bigoplus_{j=1}^k \left(\text{Hom}_{\widehat{\mathcal{O}}_x} \left(\widehat{\mathcal{E}}^{\varphi_i} \otimes \widehat{\mathbf{R}}_i, \widehat{\mathcal{E}}^{\varphi_j} \otimes \widehat{\mathbf{R}}_j \right), \widehat{\nabla} \right) \\
 & \simeq \bigoplus_{i=1}^k \bigoplus_{j=1}^k \left(\text{Hom}_{\widehat{\mathcal{O}}_x} \left(\widehat{\mathbf{R}}_i, \widehat{\mathcal{E}}^{\varphi_j - \varphi_i} \otimes \widehat{\mathbf{R}}_j \right), \widehat{\nabla} \right) \\
 & \simeq \bigoplus_{i=1}^k \bigoplus_{j=1}^k \left(\text{Hom}_{\widehat{\mathcal{O}}_x} \left(\widehat{\mathbf{R}}_i, \widehat{\mathbf{R}}_j \right) \otimes \widehat{\mathcal{E}}^{\varphi_j - \varphi_i}, \widehat{\nabla} \right),
 \end{aligned}$$

the statement i) implies that

$$\begin{aligned}
 \chi(\text{End}(\widehat{\mathcal{N}})_{\min}, \mathbb{C}[[x]]) &= \dim \{ \mathbf{M} = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_k \mid \text{ad}_{\mathbf{T}_1} \oplus \cdots \oplus \text{ad}_{\mathbf{T}_k} \mathbf{M} = \mathbf{M} \} \\
 &= \sum_{i=1}^k \dim \mathbf{Z}(\mathbf{T}_i).
 \end{aligned}$$

iii) Given the decomposition of Turruttin (1) of $(\widehat{\mathcal{N}}, \widehat{\nabla})$, where $(\widehat{\mathbf{R}}_i, \widehat{\nabla}_i)$ are regular meromorphic connexions, the irregularity of each component is either 0, if $i = 1$, or $n_i = \text{rang}(\widehat{\mathbf{R}}_i)$, if $i > 1$. Applying the same reasoning to the decomposition of Turruttin of the endomorphisms of $\widehat{\mathcal{N}}$, i.e. to the identity (10), one has:

$$i(\mathcal{E}nd_{\widehat{\mathcal{O}}}(\widehat{\mathcal{N}})_{\min}) = \sum_{\substack{i,j=1 \\ i \neq j}}^k \text{rang} \left(\text{Hom}_{\widehat{\mathcal{O}}_x}(\widehat{\mathbf{R}}_i, \widehat{\mathbf{R}}_j) \right) = \left(\sum_{i=1}^k n_i \right)^2 - \sum_{i=1}^k n_i^2,$$

where $n_i = \text{rang}(\mathbf{T}_i) = \text{rang}(\widehat{\mathbf{R}}_i)$. □

4. PAIRS OF VECTOR SPACES

The previous two sections show that both the rigidity index of a regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$ -module as well as of its Fourier transform are expressed in terms of the monodromy at its singular points, cf. theorems 2.1 and 3.3. Moreover not only the category Θ of pairs of vector spaces is equivalent (resp. anti-equivalent) to the category of regular holonomic \mathcal{D}_x -modules (resp. the category of germs of complexes of perverse sheaves), but also the monodromy appears there naturally.

Theorem 4.1. ([8], Proposition 2.4.5) *Let \mathcal{F}^\bullet be a complex of perverse sheaves on $D = B_\varepsilon(0) \subset \mathbb{C}$, $\varepsilon > 0$, $j : D^* \hookrightarrow D$, $E = h^0(\mathcal{F}^\bullet(D \setminus \mathbb{R}^+))$, $F = (\mathbf{R}^1 \Gamma_{\mathbb{R}^+ \cap D} \mathcal{F}^\bullet)_0$ and T the monodromy of the local system $h^0(\mathcal{F}^\bullet|_{D^*})$. One has the following representations in the category Θ :*

$$\text{i) If } \mathcal{F}^\bullet = j_!(h^0(\mathcal{F}^\bullet|_{D^*})), \mathcal{F}^\bullet \text{ is represented by } E \begin{array}{c} \xrightarrow{\mathbf{1}_E} \\ \xleftarrow{T - \mathbf{1}_E} \end{array} E, \text{ where } E = F.$$

$$\text{ii) If } \mathcal{F}^\bullet = j_*(h^0(\mathcal{F}^\bullet|_{D^*})), \mathcal{F}^\bullet \text{ is represented by } E \begin{array}{c} \xrightarrow{T - \mathbf{1}_E} \\ \xleftarrow{\text{inc}} \end{array} F, \text{ where } F \subset E.$$

iii) If $\mathcal{F}^\bullet = Rj_*(h^0(\mathcal{F}^\bullet|_{D^*}))$, \mathcal{F}^\bullet is represented by $E \begin{smallmatrix} \xrightarrow{T - \mathbb{1}_E} \\ \xleftarrow{\mathbb{1}_E} \end{smallmatrix} E$, where $E = F$.

Remark 4.2. The pair of vector spaces in statement iii) of theorem above is the representant in the category Θ of the germs of localized regular holonomic \mathcal{D} -modules (see [6] pp. 40 and 41).

The following Theorem shows that the notion of minimal extension in the category Θ is meaningful.

Theorem 4.3. ([8], Proposition 2.7.13) *Let D be a disk centered at the origin, $j : D^* \hookrightarrow D$ the inclusion and \mathcal{L} a locally constant sheaf on D^* . If \mathcal{F}^\bullet is a complex of perverse sheaves on D such that $\mathcal{F}^\bullet|_{D^*} = \mathcal{L}$, then:*

- i) *there exists $\eta : j_*\mathcal{L} \rightarrow \mathcal{F}^\bullet$ morphism of complexes of perverse sheaves such that $\eta|_{D^*} = id_{\mathcal{L}}$,*
- ii) *$j_*\mathcal{L}$ has neither kernels nor cokernels with support at the origin.*

Kernels, cokernels are taken in the category of complexes of perverse sheaves sense.

Definition 4.4 (Minimal extension). Let D be a disk centered at the origin, $j : D^* \hookrightarrow D$ the inclusion, \mathcal{F}^\bullet a complex of perverse sheaves on D and $\mathcal{L} = h^0(\mathcal{F}^\bullet|_{D^*})$. One calls **minimal extension** of \mathcal{F}^\bullet to the complex $j_*\mathcal{L}$.

Proposition 4.5. *If the pair $E \begin{smallmatrix} \xrightarrow{u} \\ \xleftarrow{v} \end{smallmatrix} F \neq 0 \begin{smallmatrix} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{smallmatrix} 0$ is isomorphic to its minimal extension, i.e. $E \begin{smallmatrix} \xrightarrow{v \circ u} \\ \xleftarrow{inc} \end{smallmatrix} \text{im}(v \circ u)$, then*

$$\dim Z(v \circ u) - \dim Z(u \circ v) = (\dim \ker(v \circ u))^2.$$

Proof. Cf. [8] Proposition 2.4.10. Idea: decompose in Jordan blocks. \square

To find a relationship between the terms figuring in the rigidity index in theorems 2.1 and 3.3, take the representants of \mathcal{M}_{x_i} , $x_i \in \Sigma$ in the category Θ

$$T_i - \mathbb{1} \begin{smallmatrix} \hookrightarrow \\ \hookleftarrow \end{smallmatrix} E_i \begin{smallmatrix} \xrightarrow{u_i} \\ \xleftarrow{v_i} \end{smallmatrix} F_i \begin{smallmatrix} \hookrightarrow \\ \hookleftarrow \end{smallmatrix} T_{F_i} - \mathbb{1}.$$

Since M is irreducible, M is equal to its minimal extension. By Theorem 4.1 ii) and Definition 4.4, for each $x_i \in \Sigma \cap \mathbb{C}$, the pair above is equivalent to

$$T_i - \mathbb{1} \begin{smallmatrix} \hookrightarrow \\ \hookleftarrow \end{smallmatrix} E_i \begin{smallmatrix} \xrightarrow{T_i - \mathbb{1}} \\ \xleftarrow{inc} \end{smallmatrix} F_i \begin{smallmatrix} \hookrightarrow \\ \hookleftarrow \end{smallmatrix} T_{F_i} - \mathbb{1}.$$

As M is holonomic, $M = A_1/I$. In particular if one takes a division basis (P_p, \dots, P_q) of I , then $\dim E_i = \deg_{\partial_x} P_p$. A proof can be found in [10] Theorem I.1.1. Without loss of generality $E_i = E$, so the pairs above can be rewritten as follows

$$(11) \quad T_i - \mathbb{1} \begin{smallmatrix} \hookrightarrow \\ \hookleftarrow \end{smallmatrix} E \begin{smallmatrix} \xrightarrow{T_i - \mathbb{1}} \\ \xleftarrow{inc} \end{smallmatrix} F_i \begin{smallmatrix} \hookrightarrow \\ \hookleftarrow \end{smallmatrix} T_{F_i} - \mathbb{1}.$$

On the other hand, as $\mathcal{M}_{\mathcal{F}}$ is regular at 0, $(\mathcal{M}_{\mathcal{F}})_0$ is equivalent to

$$\widehat{T} - \mathbb{1} \hookrightarrow \widehat{E} \begin{array}{c} \xrightarrow{u} \\ \xleftarrow{v} \end{array} \widehat{F} \hookrightarrow T_{\widehat{F}} - \mathbb{1}.$$

As \mathcal{M} is irreducible, $\mathcal{M}_{\mathcal{F}}$ is irreducible (because the Fourier transform is an equivalence of categories), therefore coincides with its minimal extension, so by Theorem 4.3 and Definition 4.4 the pair above is equivalent to

$$(12) \quad \widehat{T} - \mathbb{1} \hookrightarrow \widehat{E} \begin{array}{c} \xrightarrow{\widehat{T} - \mathbb{1}} \\ \xleftarrow{inc} \end{array} \widehat{F} \hookrightarrow T_{\widehat{F}} - \mathbb{1}.$$

Besides there is also the information provided by the Turrittin decomposition of the Fourier transform at infinity. Let $(\widehat{\mathcal{N}}, \widehat{\nabla})$ be the formalized of the meromorphic connexion $(\mathcal{N}, \nabla) \doteq \mathcal{M}_{\mathcal{F}}[*\{\infty\}]$. Its Turrittin decomposition is

$$(\widehat{\mathcal{N}}, \widehat{\nabla}) \simeq \bigoplus_{i=1}^k \widehat{\mathcal{E}}^{\varphi_i} \otimes (\widehat{R}_i, \widehat{\nabla}_i).$$

By Remark 4.2, each connexion $(\widehat{R}_i, \widehat{\nabla}_i)$ is equivalent to

$$\widehat{F}_i \begin{array}{c} \xrightarrow{T_{\widehat{F}_i} - \mathbb{1}} \\ \xleftarrow{\mathbb{1}} \end{array} \widehat{F}_i.$$

In [7] Malgrange shows analytically that $F_i = \widehat{F}_i$ and $\widehat{F} = E_{\infty}$, cf. [7] Theorem XII.2.9. Moreover the later equality is a corollary of this stronger result.

Lemma 4.6. *Let \mathcal{M} be a regular holonomic $\mathcal{D}_{\mathbb{P}^1}[*\{\infty\}]$ -module and $\mathcal{M}_{\mathcal{F}}$ its Fourier transform. If (E', F') (resp. $(\widehat{E}, \widehat{F})$) is the pair of vector spaces equivalent to \mathcal{M}_{∞} (resp. $(\mathcal{M}_{\mathcal{F}})_0$), then $\widehat{F} = F'$ and $T_{\widehat{F}} = T_{F'}$.*

Proof. An algebraic proof can be found in [8] Lemma 2.6.22. \square

Given all this one is led to believe that the following two lemmas are true and one proves that algebraically.

Lemma 4.7. *If $\mathcal{M}_{\min} = \mathcal{M}$, then for each $x_i \in \Sigma \cap \mathbb{C}$ the monodromies $T_{F_i} : F_i \rightarrow F_i$ and $T_{\widehat{F}_i} : \widehat{F}_i \rightarrow \widehat{F}_i$ are conjugated, i.e. \mathcal{M}_{x_i} is equivalent to*

$$T_i - \mathbb{1} \hookrightarrow E \begin{array}{c} \xrightarrow{T_i - \mathbb{1}} \\ \xleftarrow{inc} \end{array} F_i \hookrightarrow T_{\widehat{F}_i} - \mathbb{1}.$$

Lemma 4.8. *If $(\mathcal{M}_{\mathcal{F}})_{\min} = \mathcal{M}_{\mathcal{F}}$ then the monodromies at $\tau = 0$ $T_{\widehat{E}} : \widehat{E} \rightarrow \widehat{E}$ and $T_{\infty} : F_{\infty} \rightarrow F_{\infty}$ are conjugated, i.e. $(\mathcal{M}_{\mathcal{F}})_0$ is equivalent to*

$$\widehat{T}_0 - \mathbb{1} \hookrightarrow \widehat{E} \begin{array}{c} \xrightarrow{T_0 - \mathbb{1}} \\ \xleftarrow{inc} \end{array} \widehat{F} \hookrightarrow T_{\infty} - \mathbb{1}.$$

To prove Lemma 4.7, one briefly recalls the notation used by Sabbah in [9] to compute the Turrittin decomposition of $(\widehat{\mathcal{N}}, \widehat{\nabla})$ in the microlocalization context. Those notations will be used just in the following three statements.

Notation 4.9. The Fourier transform is the isomorphism of algebras:

$$\begin{aligned} \mathbb{C}[t]\langle \partial_t \rangle &\longrightarrow \mathbb{C}[\tau']\langle \partial_{\tau'} \rangle \\ t &\longmapsto -\partial_{\tau'} \\ \partial_t &\longmapsto \tau' \end{aligned}$$

denoted by $P \mapsto \widehat{P}$. Every $\mathbb{C}[t]\langle \partial_t \rangle$ -module M becomes this way a $\mathbb{C}[\tau']\langle \partial_{\tau'} \rangle$ -module denoted by \widehat{M} and called the Fourier transform of M . From now on M is assumed holonomic with regular singularities including at infinity. One knows that M only has a regular singularity at $\tau' = 0$ and a possibly irregular singularity at $\tau' = \infty$. The localized module $\widehat{M}[\tau']$ is still holonomic and is a free $\mathbb{C}[\tau', \tau'^{-1}]$ -module of finite rank. It can be regarded as a meromorphic bundle on the Riemann sphere \mathbb{P}^1 , covered by charts with coordinates τ and τ' and with the transition map $\tau = \tau'^{-1}$ on their intersection. Thus it is a free $\mathbb{C}[\tau, \tau^{-1}]$ -module. Finally one denotes by \mathcal{M} the $\mathcal{D}_{\mathbb{P}^1}$ -module $\mathcal{O}_{\mathbb{P}^1} \otimes_{\mathbb{C}[t]} M$.

Lemma 4.10. ([9], Lemma 3.3) *The microlocalized module \mathcal{M}^μ has support in the set of singular points of \mathcal{M} .*

As \mathcal{M}^μ has support at the singular points c of \mathcal{M} , one can write $\Gamma(\mathbb{C}, \mathcal{M}^\mu) = \bigoplus_c \mathcal{M}_c^\mu$.

Proposition 4.11. ([9], Proposition 3.4) *At any singular point c of \mathcal{M} , the germ $\mathcal{E}^{c/\tau} \otimes \mathcal{M}_c^\mu$ is a $(\widehat{k} \doteq \mathbb{C}[[\tau]][\tau^{-1}], \widehat{\nabla})$ -vector space with regular singularities.*

Proposition 4.12. ([9], Proposition 3.6) *The composed $\mathbb{C}[[\tau]]$ -linear mapping*

$$\widehat{\mathbb{G}} \doteq \widehat{k} \otimes_{\mathbb{C}[[\tau^{-1}]]} \widehat{M} \rightarrow \Gamma(\mathbb{C}, \widehat{k} \otimes_{\mathbb{C}[[\tau]]} \mathcal{M}) \rightarrow \Gamma(\mathbb{C}, \mathcal{M}^\mu)$$

is an isomorphism.

Proof of Lemma 4.7. Thanks to Lemma 4.10, for each $x_i \in \Sigma \cap \mathbb{C}$ the microlocalization morphism μ allows the construction of the exact sequence of holonomic \mathcal{D}_{x_i} -modules

$$(13) \quad 0 \longrightarrow \ker \mu \xrightarrow{\quad} \mathcal{M}_{x_i} \xrightarrow{\quad \mu \quad} (\mathcal{M}_{x_i})^\mu \longrightarrow \operatorname{coker} \mu \longrightarrow 0.$$

As $\ker \mu \simeq \mathcal{D}_{x_i} / \mathcal{D}_{x_i} \cdot (\partial_{x_i})^k$ (resp. $\operatorname{coker} \mu \simeq \mathcal{D}_{x_i} / \mathcal{D}_{x_i} \cdot (\partial_{x_i})^{k'}$) for a given $k \in \mathbb{N}$ (resp. $k' \in \mathbb{N}$), $\ker \mu$ (resp. $\operatorname{coker} \mu$) is equivalent to $\mathbb{C}^k \begin{smallmatrix} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{smallmatrix} 0$ (resp. $\mathbb{C}^{k'} \begin{smallmatrix} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{smallmatrix} 0$). By Proposition 4.11 $(\mathcal{M}_{x_i})^\mu$ is a regular meromorphic connexion therefore Remark 4.2 implies that it is equivalent to

$$\widehat{\mathbb{F}}_i \begin{smallmatrix} \xrightarrow{\quad T_{\widehat{\mathbb{F}}_i} - \mathbb{I} \quad} \\ \xleftarrow{\quad \mathbb{I} \quad} \end{smallmatrix} \widehat{\mathbb{F}}_i.$$

These equivalences imply that the exact sequence (13) is equivalent to the exact sequence of pairs of vector spaces

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{C}^k & \xrightarrow{\quad} & E & \xrightarrow{\quad \alpha \quad} & \widehat{F}_i \longrightarrow \mathbb{C}^{k'} \longrightarrow 0 \\
 & & \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & & j \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & \begin{array}{c} T_i - \mathbb{1} \\ \sim \\ \beta \end{array} & \mathbb{1} \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) T_{\widehat{F}_i} - \mathbb{1} \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \\
 0 & \longrightarrow & 0 & \longrightarrow & F_i & \xrightarrow{\quad \sim \quad} & \widehat{F}_i \longrightarrow 0 \longrightarrow 0.
 \end{array}$$

Since the rows are exact, β is an isomorphism, furthermore the commutativity of this diagram implies that $\alpha \circ j = \beta$ and $\beta \circ (T_i - \mathbb{1}) = (T_{\widehat{F}_i} - \mathbb{1}) \circ \alpha$, thus $\beta \circ (T_i - \mathbb{1}) \circ j = (T_{\widehat{F}_i} - \mathbb{1}) \circ \alpha \circ j$, hence $\beta \circ T_i = T_{\widehat{F}_i} \circ \beta$. \square

Corollary 4.13. $\dim Z(T_i) - \dim Z(T_{\widehat{F}_i}) = (\dim \ker(T_i - \mathbb{1}))^2 = (\dim E - \dim \widehat{F}_i)^2$

Proof. As the pair of vector spaces $(E, F_i, T_i - \mathbb{1}, j)$ is minimal, it follows from Proposition 4.5 that $\dim Z(T_i) - \dim Z(T_{\widehat{F}_i}) = (\dim \ker(T_i - \mathbb{1}))^2$. On the other hand $\dim \ker(T_i - \mathbb{1}) + \dim \operatorname{im}(T_i - \mathbb{1}) = \dim E$. Since $T_i - \mathbb{1} : E \rightarrow F_i$ is onto $\dim F_i = \dim \operatorname{im}(T_i - \mathbb{1})$, $(\dim \ker(T_i - \mathbb{1}))^2 = (\dim E - \dim \widehat{F}_i)^2$. \square

Proof of Lemma 4.8. Let us start with the exact sequence

$$0 \longrightarrow \ker \lambda \xrightarrow{\quad} \mathcal{M}_\infty \xrightarrow{\quad \lambda \quad} \mathcal{M}_\infty[*\{\infty\}] \longrightarrow \operatorname{coker} \lambda \longrightarrow 0.$$

As $\ker \lambda \simeq \mathcal{D}_{t'}/\mathcal{D}_{t'}t'^k$ (resp. $\ker \lambda \simeq \mathcal{D}_{t'}/\mathcal{D}_{t'}t'^{k'}$) for a given $k \in \mathbb{N}$ (resp. $k' \in \mathbb{N}$), the exact sequence above gives rise to the exact sequence

$$0 \longrightarrow \mathcal{D}_\tau/\mathcal{D}_\tau(\partial_\tau)^k \xrightarrow{\quad} (\mathcal{M}_\mathcal{F})_0 \longrightarrow (\mathcal{M}[t^{-1}]_\mathcal{F})_0 \longrightarrow \mathcal{D}_\tau/\mathcal{D}_\tau(\partial_\tau)^{k'} \longrightarrow 0,$$

which is equivalent to the exact sequence of pairs of vector spaces

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{C}^k & \xrightarrow{\quad} & \widehat{E} & \xrightarrow{\quad \alpha \quad} & \widehat{E}' \longrightarrow \mathbb{C}^{k'} \longrightarrow 0 \\
 & & \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & & j \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & \begin{array}{c} \widehat{T}_0 - \mathbb{1} \\ \sim \\ \beta \end{array} & v' \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) u' \\
 0 & \longrightarrow & 0 & \longrightarrow & \widehat{F} & \xrightarrow{\quad \sim \quad} & \widehat{F}' \longrightarrow 0 \longrightarrow 0.
 \end{array}$$

Since rows are exact, β is an isomorphism. Furthermore the commutativity of the diagram above implies that $u' \circ \alpha = \beta \circ (\widehat{T}_0 - \mathbb{1})$ and $v' \circ \beta = \alpha \circ j$, thus $u' \circ \alpha \circ j = \beta \circ (\widehat{T}_0 - \mathbb{1}) \circ j$, hence $(T_{\widehat{F}'} - \mathbb{1}) \circ \beta = \beta \circ (\widehat{T}_0 - \mathbb{1})$, $\beta \circ \widehat{T}_0 = T_{\widehat{F}'} \circ \beta$. Thanks to Lemma 4.6 \widehat{T}_0 and T_∞ are conjugated. \square

Corollary 4.14. $\dim Z(T_\infty) - \dim Z(\widehat{T}_0) = -(\dim \ker(\widehat{T}_0 - \mathbb{1}))^2 = -(\dim \widehat{F} - \dim F_\infty)^2$

Proof. Similar to Corollary 4.13. \square

Corollary 4.15. $\dim \ker(\widehat{T}_0 - \mathbb{1}) = \dim \widehat{E} - \dim E = \sum_{i=1}^k \dim \widehat{F}_i - \dim E$.

Proof. Thanks to Lemma 4.8 $\widehat{T}_0 - \mathbb{1} : \widehat{E} \rightarrow \widehat{F}$ is onto, therefore $\dim \ker(\widehat{T}_0 - \mathbb{1}) = \dim \widehat{E} - \dim \widehat{F}$. This Lemma also implies that $\dim \widehat{F} = \dim F_\infty = \dim E$, because \mathcal{M} is regular and localized at infinity. It follows from Proposition 4.12 that $\dim \ker(\widehat{T}_0 - \mathbb{1}) = \sum_{i=1}^k \dim \widehat{F}_i - \dim \widehat{F}$. \square

5. RIGIDITY INDEX PRESERVATION

The main result of this paper can now be proved.

Theorem 5.1. *If \mathcal{M} is a regular irreducible holonomic $\mathcal{D}_{\mathbb{P}^1}$ -module with singularities on Σ and localized at infinity, such that $\mathcal{M}_{\min} \neq 0$, then the Fourier transform preserves the rigidity index.*

Proof. The irreducibility condition ensures that $\mathcal{M}_{\min} = \mathcal{M}$, unless $H_{[\Sigma]}(\mathcal{M}) = \mathcal{M}$, cf. Proposition 2.5. It follows from Theorem 3.3 that:

$$\text{rig}(\mathcal{M}_{\mathcal{F}}) = \dim Z(\widehat{T}_0) + \sum_{i=1}^k \dim Z(\widehat{T}_i) + \sum_{i=1}^k (\dim \widehat{F}_i)^2 - (\dim \widehat{E})^2.$$

Thanks to corollaries 4.13 and 4.14 the equality above can be rewritten as follows:

$$\begin{aligned} \text{rig}(\mathcal{M}_{\mathcal{F}}) &= \dim Z(T_{\infty}) + (\dim \ker(\widehat{T}_0 - \mathbb{1}))^2 + \\ &+ \sum_{i=1}^k [\dim Z(T_i) - (\dim \ker(T_i - \mathbb{1}))^2] + \sum_{i=1}^k (\dim \widehat{F}_i)^2 - (\dim \widehat{E})^2. \end{aligned}$$

Moreover by corollaries 4.15 and 4.13

$$\begin{aligned} \text{rig}(\mathcal{M}_{\mathcal{F}}) &= \dim Z(T_{\infty}) + (\dim \widehat{E} - \dim E)^2 + \\ &+ \sum_{i=1}^k [\dim Z(T_i) - (\dim E - \dim \widehat{F}_i)^2] + \sum_{i=1}^k (\dim \widehat{F}_i)^2 - (\dim \widehat{E})^2 \\ &= \dim Z(T_{\infty}) + \sum_{i=1}^k \dim Z(T_i) + (\dim \widehat{E} - \dim E)^2 - \\ &- \sum_{i=1}^k [(\dim E)^2 - 2 \dim E \dim \widehat{F}_i + (\dim \widehat{F}_i)^2] + \sum_{i=1}^k (\dim \widehat{F}_i)^2 - (\dim \widehat{E})^2 \\ &= \dim Z(T_{\infty}) + \sum_{i=1}^k \dim Z(T_i) + (\dim \widehat{E} - \dim E)^2 - \\ &- k(\dim E)^2 + 2 \dim E \sum_{i=1}^k \dim \widehat{F}_i - (\dim \widehat{E})^2 \\ &= \dim Z(T_{\infty}) + \sum_{i=1}^k \dim Z(T_i) + (\dim \widehat{E})^2 - 2 \dim \widehat{E} \dim E + \\ &+ (\dim E)^2 - k(\dim E)^2 + 2 \dim E \dim \widehat{E} - (\dim \widehat{E})^2 \\ &= (2 - (k+1))(\dim E)^2 + \dim Z(T_{\infty}) + \sum_{i=1}^k \dim Z(T_i) \\ &= \text{rig}(\mathcal{M}), \end{aligned}$$

therefore the Fourier transform preserves the rigidity index. \square

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